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1999 J. Phys. A: Math. Gen. 32 3485

(http://iopscience.iop.org/0305-4470/32/19/302)

J. Phys. A: Math. Gen. **32** (1999) 3485–3495. Printed in the UK PII: S0305-4470(99)98619-5

Dynamical q-deformation in quantum theory and the stochastic limit

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Received 20 October 1998, in final form 19 February 1999

Abstract. We show that in a model of a particle interacting with a quantum field, the field operators rescaled according to the prescriptions of the stochastic limit, obey q-commutational relations with q depending on time. After the stochastic limit, due to the nonlinearity, the particle and field degrees of freedom become *entangled* even at a kinematical level in the sense that the field and the atomic variables no longer commute but give rise to a new algebra with new commutation relations replacing the boson ones. This allows to give a simple proof of the fact that the non-crossing half-planar diagrams give the dominating contribution in a weak-coupling regime and to calculate explicitly the correlations associated with the new algebra.

1. Introduction

In recent years there has been a great interest in q-deformed commutational relations, see for example [1–6]. In many works q-deformed relations are considered as an *ad hoc* deformation of the ordinary commutation relations or as a hidden symmetry algebra.

In this work we show that the so-called collective operators $a_\lambda(t,k)$ satisfy the q-deformed commutation relations (see equation (14) below), where the parameter q depends on time. The collective operators are natural objects in the stochastic (van Hove) limit of the model describing interaction of a particle with a quantum field. The stochastic limit is used to derive the long-time behaviour of the system interacting with a reservoir, in particular, to derive the master equation [7, 8]. The main result of this work is that in the stochastic limit the q-deformed commutation relations give rise to the generalized quantum Boltzmann commutational relations.

We investigate a model describing the interaction of non-relativistic particle with a quantum field. This model is widely studied in elementary particle physics, solid state physics, quantum optics, see for example [9–12]. We consider the simplest case in which matter is represented by a single particle or quasiparticle whose position and momentum we denote, respectively, by $q=(q_1,q_2,q_3)$ and $p=(p_1,p_2,p_3)$ and satisfy the commutation relations $[q_h,p_k]=\mathrm{i}\delta_{hk}$. The quantum field is described by boson operators $a(k)=(a_1(k),a_2(k),a_3(k))$; $a^\dagger(k)=(a_1^\dagger(k),\dots,a_3^\dagger(k))$ satisfying the *canonical commutation*

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relations $[a_j(k), a_h^\dagger(k')] = \delta_{jh}\delta(k-k')$. The Hamiltonian of a non-relativistic particle interacting with a quantum field is, neglecting polarization,

$$H = H_0 + \lambda H_I = \int \omega(k)\, a^\dagger(k)\, a(k)\, \mathrm{d}k + \tfrac{1}{2}p^2 + \lambda H_I \tag{1}$$

where λ is a small constant, $\omega(k)$ is a dispersion law and

$$H_I = p\cdot\mathcal{A}(q) + \mathcal{A}(q)\cdot p := \int \mathrm{d}^3k\, (g(k)p\cdot \mathrm{e}^{\mathrm{i}kq}a^\dagger(k) + \overline{g}(k)p\cdot \mathrm{e}^{-\mathrm{i}kq}a(k)) + \text{h.c.} \tag{2}$$

The general idea of the stochastic limit is to make the time rescaling $t \to t/\lambda^2$ in the solution of the Schrödinger equation in the interaction picture $U_t^{(\lambda)} = \mathrm{e}^{\mathrm{i}tH_0}\mathrm{e}^{-\mathrm{i}tH}$, associated to the Hamiltonian H, i.e.

$$\frac{\partial}{\partial t}U_t^{(\lambda)} = -\mathrm{i}\lambda H_I(t)\, U_t^{(\lambda)} \qquad U_0^{(\lambda)} = 1 \tag{3}$$

with $H_I(t) = \mathrm{e}^{\mathrm{i}tH_0}H_I\mathrm{e}^{-\mathrm{i}tH_0}$ (the *evolved interaction Hamiltonian*). This leads to the rescaled equation

$$\frac{\partial}{\partial t}U_{t/\lambda^2}^{(\lambda)} = -\frac{\mathrm{i}}{\lambda}H_I(t/\lambda^2)\, U_{t/\lambda^2}^{(\lambda)} \tag{4}$$

and one wants to study the limits, in a topology to be specified,

$$\lim_{\lambda\to 0} U_{t/\lambda^2}^{(\lambda)} = U_t \tag{5}$$

$$\lim_{\lambda\to 0}\frac{1}{\lambda}H_I\left(\frac{t}{\lambda^2}\right) = H_t = \int \mathrm{d}^3k\, \big(g(k)p\cdot b^\dagger(t,k) + \overline{g}(k)p\cdot b(t,k) + \text{h.c.}\big). \tag{6}$$

Moreover, one wants to prove that U_t is the solution of the equation

$$\partial_t U_t = -\mathrm{i}H_tU_t \qquad U_0 = 1. \tag{7}$$

The interest of this limit equation is in the fact that many problems become explicitly integrable. The stochastic limit of the model (1) and (2) has been considered in [7, 8, 13–15]. In this work we use a new method. We show that the field operators satisfy q-commutational relations and use this fact to compute the stochastic limit for correlation functions and to establish the new algebra.

The rescaling $t \to t/\lambda^2$ is equivalent to considering the simultaneous limit $\lambda \to 0$, $t\to\infty$ under the condition that $\lambda^2 t$ tends to a constant (interpreted as a new *slow scale* time). This limit captures the main contributions to the dynamics in a regime, of *long times and small coupling* arising from the cumulative effects, on a large time scale, of small interactions ($\lambda \to 0$). The physical idea is that, looked at from the slow time scale of the atom, the field looks like a very chaotic object: a *quantum white noise*, i.e. a δ-correlated (in time) quantum field $b^\dagger(t,k)$, $b(t,k)$ also called a *master field*. If one introduces the dipole approximation the master field is the usual boson Fock white noise. Without the dipole approximation the master field is a completely new type of white noise whose algebra is described by the relations [8]

$$b(t,k)p = (p+k)b(t,k) \tag{8}$$

$$b(t,k)\, b^\dagger(t',k') = 2\pi\delta(t-t')\,\delta(\tilde\omega(k)+kp)\,\delta(k-k') \tag{9}$$

$$\tilde\omega(k) := \omega(k) + \tfrac{1}{2}k^2. \tag{10}$$

Recalling that p is the atomic momentum, we see that the relation (8) shows that the atom and the master field are not independent even at a kinematical level. This is what we call *entanglement*. The relation (9) is a generalization of the algebra of free creation–annihilation operators with commutation relations

$$A_i A_j^\dagger = \delta_{ij}$$

and the corresponding statistics becomes a generalization of the Boltzmannian (or free) statistics. This generalization is due to the fact that the right-hand side is not a scalar but an operator (a function of the atomic momentum). This means that the relations (8) and (9) are *module commutation relations*.

For any fixed value $\bar{p}$ of the atomic momentum we obtain a copy of the free (or Boltzmannian) algebra. Given the relations (8)–(10), the statistics of the master field is uniquely determined by the condition

$$b(t,k)\Psi = 0 \tag{11}$$

where Ψ is the vacuum of the master field, via a module generalization of the free Wick theorem (this is our theorem 2 in section 4 below).

In section 2 the dynamically q-deformed commutation relations are obtained and the stochastic limit for collective operators is evaluated. In section 3 the n-point correlation functions of the collective operators are computed. Finally, in section 4 the stochastic limit of n-point correlation functions is calculated.

2. Dynamical q-deformation

In order to determine the limit (3) one rewrites the rescaled interaction Hamiltonian in terms of the rescaled fields $a_\lambda(t,k)$:

$$\frac{1}{\lambda} H_I\left(\frac{t}{\lambda^2}\right) = \frac{1}{\lambda} A(t/\lambda^2) + \text{h.c.} = \int \mathrm{d}^3k\, p(\overline{g}(k)\, a_\lambda(t,k) + g(k)\, a_\lambda^\dagger(t,k)) + \text{h.c.} \tag{12}$$

The algebra of the rescaled fields in the stochastic limit will give rise to the algebra of the master field. Using the standard commutation relation $[p,q] = -\mathrm{i}$ we obtain the rescaled interaction

$$a_\lambda(t,k) := \frac{1}{\lambda}\, \mathrm{e}^{\mathrm{i}(t/\lambda^2)\, H_0} \mathrm{e}^{-\mathrm{i}kq} a(k)\, \mathrm{e}^{-\mathrm{i}(t/\lambda^2)\, H_0} = \frac{1}{\lambda} \mathrm{e}^{-\mathrm{i}(t/\lambda^2)\,(\tilde{\omega}(k)+kp)} \mathrm{e}^{-\mathrm{i}kq} a(k) \tag{13}$$

where $\tilde{\omega}(k) = \omega(k) + \frac{1}{2}k^2$.

It is now easy to prove that operators $a_\lambda(t,k)$ satisfy the following q-deformed module relations,

$$a_\lambda(t,k)\, a_\lambda^\dagger(t',k') = a_\lambda^\dagger(t',k')\, a_\lambda(t,k) \cdot q_\lambda(t-t', kk') + \frac{1}{\lambda^2}\, q_\lambda(t-t', \tilde{\omega}(k)+kp)\, \delta(k-k') \tag{14}$$

$$a_\lambda(t,k)p = (p+k)\, a_\lambda(t,k) \tag{15}$$

where

$$q_\lambda(t-t',x) = \mathrm{e}^{-\mathrm{i}(t-t'/\lambda^2)\, x} \tag{16}$$

is an oscillating exponent.

This shows that the module q-deformation of the commutation relations arises here as a result of the dynamics and are not put artificially *ab initio*. Now let us suppose that the master field

$$b(t,k) = \lim_{\lambda\to 0} a_\lambda(t,k) \tag{17}$$

exists. Then it is natural to conjecture that its algebra shall be obtained as the stochastic limit ($\lambda \to 0$) of the algebra (14) and (15). Notice that the factor $q_\lambda(t-t', x)$ is an oscillating exponent and one easily sees that

$$\lim_{\lambda\to 0} q_\lambda(t,x) = 0 \qquad \lim_{\lambda\to 0} \frac{1}{\lambda^2}\, q_\lambda(t,x) = 2\pi\,\delta(t)\,\delta(x). \tag{18}$$

Then it is natural to expect that the limit of (15) is

$$b(t,k)\,p = (p+k)\,b(t,k) \tag{19}$$

and the limit of (14) gives the module free relation

$$b(t,k)\,b^\dagger(t',k') = 2\pi\,\delta(t-t')\,\delta(\tilde\omega(k)+kp)\delta(k-k'). \tag{20}$$

Operators $a_\lambda(t,k)$ also obey the relation

$$a_\lambda(t_1,k_1)\,a_\lambda(t_2,k_2) = a_\lambda(t_2,k_2)\,a_\lambda(t_1,k_1)\,q_\lambda^{-1}(t_1-t_2,k_1k_2). \tag{21}$$

If we will work formally then the formal limit $\lambda \to 0$ of the left-hand side of this equation is $b(t_1,k_1)\,b(t_2,k_2)$ and of the right-hand side is $b(t_2,k_2)\,b(t_1,k_1)$ times zero. However, we will prove that, in fact, the limit of the right-hand side of (21) is the same as of the left-hand side (limit of a product does not equal to the product of limits). Therefore, the limit of (21) is the trivial identity

$$b(t_1,k_1)\,b(t_2,k_2) = b(t_1,k_1)\,b(t_2,k_2). \tag{22}$$

An accurate proof that the relation (21) leads to (22) looks as follows. Let us consider for example the four-point correlator

$$\langle a_\lambda(t_1,k_1)\,a_\lambda(t_2,k_2)\,a_\lambda^\dagger(t_2',k_2')\,a_\lambda^\dagger(t_1',k_1')\rangle. \tag{23}$$

This corresponds to taking the matrix element of (21) between vacuum and two creation operators. According to (14) this correlator is equal to the sum of two terms

$$\begin{aligned}
&\frac{1}{\lambda^2}\,q_\lambda(t_2-t_2',\tilde\omega(k_2)+k_2(p+k_1))\,\delta(k_2-k_2')\frac{1}{\lambda^2}\,q_\lambda(t_1-t_1',\tilde\omega(k_1)+k_1p)\,\delta(k_1-k_1')\\
&\qquad\times q_\lambda(t_2-t_2',k_2k_2') + \frac{1}{\lambda^2}\,q_\lambda(t_1-t_2',\tilde\omega(k_1)+k_1p)\,\delta(k_1-k_2')\\
&\qquad\times\frac{1}{\lambda^2}\,q_\lambda(t_2-t_1',\tilde\omega(k_2)+k_2p)\,\delta(k_2-k_1')\,q_\lambda(t_2-t_2',k_2k_2').
\end{aligned}$$

In the stochastic limit $\lambda \to 0$ only the first term survives. The second term vanishes because $\lim_{\lambda\to 0} q_\lambda(t_2-t_2',k_2k_2') = 0$. The first term survives because

$$\begin{aligned}
&\frac{1}{\lambda^2}\,q_\lambda(t_2-t_2',\tilde\omega(k_2)+k_2(p+k_1))\,q_\lambda(t_2-t_2',k_2k_2')\\
&\qquad= \frac{1}{\lambda^2}\,q_\lambda(t_2-t_2',\tilde\omega(k_2)+k_2(p+k_1)+k_2k_2').
\end{aligned}$$

Let us now consider the behaviour of the relation (21) in the stochastic limit. According to (21) the correlator (23) is equal to

$$\begin{aligned}
&\langle a_\lambda(t_2,k_2)\,a_\lambda(t_1,k_1)\,a_\lambda^\dagger(t_2',k_2')\,a_\lambda^\dagger(t_1',k_1')\rangle q_\lambda^{-1}(t_2-t_2',k_2k_2')\\
&\qquad= q_\lambda^{-1}(t_2-t_2',k_2k_2')\biggl(\frac{1}{\lambda^2}\,q_\lambda(t_1-t_2',\tilde\omega(k_1)+k_1(p+k_1))\,\delta(k_1-k_2')\\
&\qquad\times\frac{1}{\lambda^2}\,q_\lambda(t_2-t_1',\tilde\omega(k_2)+k_2p)\,\delta(k_2-k_1')\,q_\lambda(t_1-t_2',k_1k_2')\\
&\qquad+\frac{1}{\lambda^2}\,q_\lambda(t_2-t_2',\tilde\omega(k_2)+k_2p)\,\delta(k_2-k_2')\frac{1}{\lambda^2}\,q_\lambda(t_1-t_1',\tilde\omega(k_1)+k_1p)\\
&\qquad\times\delta(k_1-k_1')\,q_\lambda(t_1-t_2',k_1k_2')\biggr).
\end{aligned}$$

We see that due to the term $q_\lambda^{-1}(t_2 - t_2', k_2k_2')$ only the second component of this correlator survives after the stochastic limit. Therefore, the stochastic limit of

$$\langle a_\lambda(t_2, k_2)\, a_\lambda(t_1, k_1)\, a_\lambda^\dagger(t_2', k_2')\, a_\lambda^\dagger(t_1', k_1')\rangle q_\lambda^{-1}(t_2 - t_2', k_2k_2')$$

does not equal to the product of the limit of the correlator and the limit of $q_\lambda^{-1}(t_2 - t_2', k_2k_2')$ (that is equal to zero). We have proved that at least for a four-point correlation function the identity (22) is the stochastic limit of the relation (21).

To finish the proof we have to prove the existence of the stochastic limit of n-point correlators. This is the subject of the next section.

3. The stochastic limit of an N-point correlator

In this section we prove the existence of the limit of the q-deformed correlators

$$\langle a_\lambda^{\varepsilon_1}(t_1, k_1)\dots a_\lambda^{\varepsilon_N}(t_N, k_N)\rangle \tag{24}$$

where a^ε means either a or $a^\dagger$ ($\epsilon = 0$ for a, $\epsilon = 1$ for $a^\dagger$) and $\langle\cdot\rangle$ denotes vacuum expectation. We also will prove that the limit of this correlator will be equal to the corresponding correlator of the master field:

$$\langle b^{\varepsilon_1}(t_1, k_1)\dots b^{\varepsilon_N}(t_N, k_N)\rangle. \tag{25}$$

Let us enumerate annihilators in the product $a_\lambda^{\varepsilon_1}(t_1, k_1)\dots a_\lambda^{\varepsilon_N}(t_N, k_N)$ as $a_\lambda(t_{m_j}, k_{m_j})$, $j = 1, \dots, J$, and enumerate creators as $a_\lambda^\dagger(t_{m_j'}, k_{m_j'})$, $j = 1, \dots, I$, $I + J = N$. This means that if $\varepsilon_m = 0$ then $a_\lambda^{\varepsilon_m}(t_m, k_m) = a_\lambda(t_{m_j}, k_{m_j})$ for $m = m_j$ (and the analogous condition for $\varepsilon_m = 1$).

Let us prove the following theorem.

Theorem 1. *The stochastic limit of a dynamically q-deformed correlator exists. Moreover,*

(a) if the number of creators is not equal to the number of annihilators, then the correlator (24) is equal to zero (even before the limit);

(b) if the number of creators is equal to the number of annihilators ($N = 2n$), then the limit is equal to the following:

$$\prod_{h=1}^n \delta(k_{m_h} - k_{m_h'})2\pi\delta(t_{m_h} - t_{m_h'})\,\delta\left(\tilde\omega(k_{m_h}) + k_{m_h}p + \sum_{\alpha: m_\alpha < m_h < m_\alpha'} k_{m_\alpha}\cdot k_{m_h}\right) \tag{26}$$

where $\{(m_j < m_j') : j = 1, \dots, n\}$ is the unique non-crossing partition of $\{1, \dots, 2n\}$ associated with $\varepsilon = \{\varepsilon_1 \dots \varepsilon_N\}$. Here non-crossing partition means that for arbitrary two pairs $(m_i < m_i')$, $(m_j < m_j')$ we have only the following possibilities: $m_i < m_i' < m_j < m_j'$, $m_j < m_j' < m_i < m_i'$, $m_i < m_j < m_j' < m_i'$, $m_j < m_i < m_i' < m_j'$; which means that the corresponding Wick diagram is non-crossing.

Proof. The proof of this theorem is by induction. We need to prove (26) for $N = 2n$. For $n = 1$ this relation is clear.

Let us assume (26) for $N = 2n - 2$ and prove that the same is true for $N = 2n$.

Let m_n be the first annihilation index in (24) starting from the right. Then (24) is equal to

$$\langle a_\lambda(t_{m_1}, k_{m_1})\cdots a_\lambda(t_{m_n}, k_{m_n})\, a_\lambda^\dagger(t_{m_n+1}, k_{m_n+1})\dots a_\lambda^\dagger(t_{m_n'}, k_{m_n'})\rangle.$$

Using the commutation relation (14) this is equal to

$$q_\lambda(t_{m_n}-t_{m_n+1},k_{m_n}k_{m_n+1})\langle a_\lambda(t_{m_1},k_{m_1})\cdots a^\dagger_\lambda(t_{m_n+1},k_{m_n+1})\,a_\lambda(t_{m_n},k_{m_n})\,a^\dagger_\lambda(t_{m_n+2},k_{m_n+2})\ldots\rangle$$
$$+\frac{1}{\lambda^2}\,\delta(k_{m_n}-k_{m_n+1})\langle a_\lambda(t_{m_1},k_{m_1})\cdots q_\lambda(t_{m_n}-t_{m_{n+1}},\tilde{\omega}(k_{m_n})+k_{m_n}p)\ldots\rangle. \quad (27)$$

Using the adjoint of the commutation relation (15), we commute $q_\lambda(t_{m_n}-t_{m_n+1},\tilde{\omega}(k)+k_{m_n}p)$ with all the annihilators on its right. Doing so the second term in (27) becomes

$$\langle a_\lambda(t_{m_1},k_{m_1})\cdots\hat{a}_\lambda(t_{m_n},k_{m_n})\,\hat{a}^\dagger_\lambda(t_{m_n+1},k_{m_n+1})\ldots a^\dagger_\lambda(t_{m'_n},k_{m'_n})\rangle$$
$$\times\delta(k_{m_n}-k_{m_n+1})\frac{1}{\lambda^2}\,q_\lambda\left(t_{m_n}-t_{m_{n+1}},\tilde{\omega}(k_{m_n})+k_{m_n}\left(p+\sum_{m'_j>m_n+1}k_{m'_j}\right)\right).$$

Here $\hat{a}$ means that we omit this operator in the product. Iterating this procedure (moving $a_\lambda(t_{m_n},k_{m_n})$ further to the right) we find that

$$\langle a_\lambda(t_{m_1},k_{m_1})\cdots a^\dagger_\lambda(t_{m'_n},k_{m'_n})\rangle$$
$$=\sum_{m'_\beta>m_n}\langle a_\lambda(t_{m_1},k_{m_1})\cdots\hat{a}_\lambda(t_{m_n},k_{m_n})\cdots\hat{a}^\dagger_\lambda(t_{m'_\beta},k_{m'_\beta})\cdots a^\dagger_\lambda(t_{m'_n},k_{m'_n})\rangle$$
$$\times\prod_{m_n<m'_j<m'_\beta}q_\lambda(t_{m_n}-t_{m'_j},k_{m_n}k_{m'_j})$$
$$\times\delta(k_{m_n}-k_{m_n+1})\frac{1}{\lambda^2}\,q_\lambda\left(t_{m_n}-t_{m'_\beta},\tilde{\omega}(k_{m_n})+k_{m_n}\left(p+\sum_{m'_\delta>m'_\beta}k_{m'_\delta}\right)\right)$$

where we make the convention that the product

$$\prod_{m_n<m'_j<m'_\beta}q_\lambda(t_{m_n}-t_{m'_j},k_{m_n}k_{m'_j}) \quad (28)$$

is equal to 1 if $\beta=j$.

Taking the stochastic limit of this recurrent relation we see that if the product (28) is non-trivial (not equal to 1) then the stochastic limit will be equal to zero. Using the induction assumption we find the recurrent relation for the stochastic limit of the correlator (24),

$$\lim_{\lambda\to0}\langle a_\lambda(t_{m_1},k_{m_1})\cdots a^\dagger_\lambda(t_{m'_n},k_{m'_n})\rangle$$
$$=\lim_{\lambda\to0}\langle a_\lambda(t_{m_1},k_{m_1})\cdots\hat{a}_\lambda(t_{m_n},k_{m_n})\,\hat{a}^\dagger_\lambda(t_{m_n+1},k_{m_n+1})\cdots a^\dagger_\lambda(t_{m'_n},k_{m'_n})\rangle$$
$$\times\delta(k_{m_n}-k_{m_n+1})\frac{1}{\lambda^2}\,q_\lambda\left(t_{m_n}-t_{m'_\beta},\tilde{\omega}(k_{m_n})+k_{m_n}\left(p+\sum_{m'_\delta>m'_\beta}k_{m'_\delta}\right)\right).$$

Taking the stochastic limit and using this recurrent relation we find the statement of the theorem. □

Theorem 2. *The correlators for the master field satisfying relations (8) and (9) with* $\langle\cdot\rangle$ *equal to the vacuum expectation in the free Fock space are equal to the stochastic limit of the corresponding correlators for dynamically* q*-deformed field (calculated in the previous theorem).*

Proof. The proof is done by computing the correlation functions using the commutation relations listed above. We investigate the correlator

$$\langle b^{\epsilon_1}(t_1,k_1)\,b^{\epsilon_2}(t_2,k_2)\ldots b^{\epsilon_N}(t_N,k_N)\rangle.$$

At first we simplify this correlator using (20). Obtained δ-functions we will move through $b^{\epsilon}(t,k)$, using (19). We will iterate this procedure until the monomial will take normally ordered form. Because the functional $\langle\cdot\rangle$ is equal to vacuum expectation, only δ-functions will survive.

The correlation function $b(t_{m'_h},k_{m'_h})\,b^{\dagger}(t_{m_h},k_{m_h})$ equals

$$\delta(k_{m'_h}-k_{m_h})2\pi\delta(t_{m'_h}-t_{m_h})\delta(\tilde{\omega}(k_{m_h})+k_{m_h}p) \tag{29}$$

and the relation (19) gives the term $\sum_{\alpha:m_\alpha<m_h<m'_\alpha}k_{m_\alpha}\cdot k_{m_h}$ in the phase shift (an argument of the last δ-function in (26)), arising from moving this δ-function through $b^{\epsilon}(t,k)$.

This finishes the proof of the theorem. □

4. Calculation of the n-point correlator

In this section we will find the exact form of the q-deformed correlators (before the stochastic limit)

$$\langle a_\lambda^{\varepsilon_1}(t_1,k_1)\ldots a_\lambda^{\varepsilon_N}(t_N,k_N)\rangle. \tag{30}$$

Let us prove the following lemma.

Lemma 1.

$$\begin{aligned}
a_\lambda(t,k)\,a_\lambda^{\varepsilon_1}(t_1,k_1)\ldots a_\lambda^{\varepsilon_N}(t_N,k_N)-\prod_{i=1}^{I}q_\lambda^{-1}(t-t_{m_i},kk_{m_i})\\
\times\prod_{j=1}^{J}q_\lambda(t-t_{m'_j},kk_{m'_j})\,a_\lambda^{\varepsilon_1}(t_1,k_1)\ldots a_\lambda^{\varepsilon_N}(t_N,k_N)\,a_\lambda(t,k)\\
=\sum_{j=1}^{I}\delta(k-k_{m'_j})\frac{1}{\lambda^2}q_\lambda(t-t_{m'_j},\tilde{\omega}(k)+kp)\prod_{m_i<m'_j}q_\lambda(t-t_{m'_j},kk_{m_i})\\
\times\prod_{m'_i<m'_j}q_\lambda^{-1}(t-t_{m'_j},kk_{m'_i})\prod_{m_i<m'_j}q_\lambda^{-1}(t-t_{m_i},kk_{m_i})\\
\times\prod_{m'_i<m'_j}q_\lambda(t-t_{m'_i},kk_{m'_i})\,a_\lambda^{\varepsilon_1}(t_1,k_1)\ldots\hat{a}_\lambda^{\dagger}(t_{m'_j},k_{m'_j})\ldots a_\lambda^{\varepsilon_N}(t_N,k_N).
\end{aligned} \tag{31}$$

Here the notion $\hat{a}_\lambda^{\dagger}$ means that we omit the operator $a_\lambda^{\dagger}$ in this product.

Proof. The proof of this lemma is by induction over N. The first step of induction is the relation (14) or (21). Given equation (31) for N, we will prove this formula for $N+1$. We consider two cases.

(a) The first case: $\varepsilon_{N+1}=0$. In this case using (31) for N and (21) we obtain

$$\begin{aligned}
a_\lambda(t,k)\,a_\lambda^{\varepsilon_1}(t_1,k_1)\ldots a_\lambda^{\varepsilon_N}(t_{N+1},k_{N+1})-q_\lambda^{-1}(t-t_{N+1},kk_{N+1})\prod_{i=1}^{I}q_\lambda^{-1}(t-t_{m_i},kk_{m_i})\\
\times\prod_{j=1}^{J}q_\lambda(t-t_{m'_j},kk_{m'_j})\,a_\lambda^{\varepsilon_1}(t_1,k_1)\ldots a_\lambda(t_{N+1},k_{N+1})\,a_\lambda(t,k)
\end{aligned}$$

$$= \sum_{j=1}^{I} \delta(k - k_{m'_j}) \frac{1}{\lambda^2} q_\lambda(t - t_{m'_j}, \tilde{\omega}(k) + kp) \prod_{m_i < m'_j} q_\lambda(t - t_{m'_j}, kk_{m_i})$$
$$\times \prod_{m'_i < m'_j} q_\lambda^{-1}(t - t_{m'_j}, kk_{m'_i}) \prod_{m_i < m'_j} q_\lambda^{-1}(t - t_{m_i}, kk_{m_i})$$
$$\times \prod_{m'_i < m'_j} q_\lambda(t - t_{m'_i}, kk_{m'_i})\, a_\lambda^{\varepsilon_1}(t_1, k_1) \dots \hat{a}_\lambda^{\dagger}(t_{m'_j}, k_{m'_j}) \dots a_\lambda^{\varepsilon_N}(t_N, k_N)$$

that is exactly (31) for $N + 1$.

(b) The second case: $\varepsilon_{N+1} = 1$. In this case using (31) for N and (14) we obtain

$$a_\lambda(t, k)\, a_\lambda^{\varepsilon_1}(t_1, k_1) \dots a_\lambda^{\varepsilon_N}(t_{N+1}, k_{N+1}) - \prod_{i=1}^{I} q_\lambda^{-1}(t - t_{m_i}, kk_{m_i})$$
$$\times \prod_{j=1}^{J} q_\lambda(t - t_{m'_j}, kk_{m'_j})\, a_\lambda^{\varepsilon_1}(t_1, k_1) \dots a_\lambda^{\varepsilon_N}(t_N, k_N)$$
$$\times \Big(a_\lambda^{\dagger}(t_{N+1}, k_{N+1})\, a_\lambda(t, k) q_\lambda(t - t_{N+1}, kk_{N+1})$$
$$+ \delta(k - k_{N+1}) \frac{1}{\lambda^2} q_\lambda(t - t_{N+1}, \tilde{\omega}(k) + kp) \Big)$$
$$= \sum_{j=1}^{I} \delta(k - k_{m'_j}) \frac{1}{\lambda^2} q_\lambda(t - t_{m'_j}, \tilde{\omega}(k) + kp) \prod_{m_i < m'_j} q_\lambda(t - t_{m'_j}, kk_{m_i})$$
$$\times \prod_{m'_i < m'_j} q_\lambda^{-1}(t - t_{m'_j}, kk_{m'_i}) \prod_{m_i < m'_j} q_\lambda^{-1}(t - t_{m_i}, kk_{m_i})$$
$$\times \prod_{m'_i < m'_j} q_\lambda(t - t_{m'_i}, kk_{m'_i})\, a_\lambda^{\varepsilon_1}(t_1, k_1) \dots \hat{a}_\lambda^{\dagger}(t_{m'_j}, k_{m'_j}) \dots a_\lambda^{\varepsilon_N}(t_N, k_N)$$
$$\times a_\lambda^{\dagger}(t_{N+1}, k_{N+1}).$$

Moving the term

$$\delta(k - k_{N+1}) \frac{1}{\lambda^2} q_\lambda(t - t_{N+1}, \tilde{\omega}(k) + kp)$$

to the right-hand side of this formula and commuting it with creators and annihilators using (15) we obtain (31) for $N + 1$. This finishes the proof of the lemma. □

The next theorem describes the form of N-point correlator.

Theorem 3.

(a) If the number of creators is not equal to the number of annihilators, then the correlator (24) is equal to zero.

(b) If the number of creators is equal to the number of annihilators ($N = 2n$), then the correlation function is equal to the following sum over pair partitions:

$$\sum_{\sigma(\varepsilon)} \prod_{h=1}^{n} \delta(k_{m_h} - k_{m'_h}) \frac{1}{\lambda^2} q_\lambda \left((t_{m_h} - t_{m'_h}), \left(\tilde{\omega}(k_{m_h}) + k_{m_h} p + \sum_{m_\alpha < m_h < m'_\alpha} k_{m_\alpha} \cdot k_{m_h} \right) \right)$$
$$\times \prod_{(m_j, m'_j), (m_i, m'_i); i, j = 1, \dots, n : m_j < m_i < m'_j < m'_i} q_\lambda(t_{m_i} - t_{m'_j}, k_{m_i} \cdot k_{m_j}) \qquad (32)$$

where $\sigma(\varepsilon) = \{(m_j < m'_j) : j = 1, \ldots, n\}$ *is a partition of* $\{1, \ldots, 2n\}$ *associated with* $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{2n})$.

Proof. The proof of this theorem is by induction over n. The first step of induction is obvious. Let us assume the correlator (24) is expressed by equation (32) for $N = 2n - 2$ and prove that the same is true for $N = 2n$. We consider the $2n$-point correlator

$$\langle a_\lambda^{\varepsilon_1}(t_1, k_1) \ldots a_\lambda^{\varepsilon_{2n}}(t_{2n}, k_{2n})\rangle.$$

It is easy to see that if this correlator is not equal to zero then the first operator is an annihilator and the last is a creator. Without loss of generality we can consider the case when the correlator is as follows:

$$\langle a_\lambda(t_{m_1}, k_{m_1})\, a_\lambda^{\varepsilon_2}(t_2, k_2) \ldots a_\lambda^{\varepsilon_{2n-1}}(t_{2n-1}, k_{2n-1})\, a_\lambda^\dagger(t_{m'_n}, k_{m'_n})\rangle. \tag{33}$$

From lemma 1 follows the following formula for this correlator:

$$\begin{aligned}(33) = {} & \sum_{j=1}^{n} \delta(k_{m_1} - k_{m'_j}) \frac{1}{\lambda^2} q_\lambda(t_{m_1} - t_{m'_j}, \tilde{\omega}(k_{m_1}) + k_{m_1} p) \\ & \times \prod_{m_i < m'_j < m'_i} q_\lambda(t_{m_1} - t_{m'_j}, k_{m_1} k_{m_i}) \prod_{m_i < m'_j} q_\lambda^{-1}(t_{m_1} - t_{m_i}, k_{m_1} k_{m_i}) \\ & \times \prod_{m'_i < m'_j} q_\lambda(t_{m_1} - t_{m'_i}, k_{m_1} k_{m'_i})\, \langle \hat{a}_\lambda(t_{m_1}, k_{m_1}) \ldots \hat{a}_\lambda^\dagger(t_{m'_j}, k_{m'_j}) \ldots\rangle. \end{aligned} \tag{34}$$

The product $\prod_{m_i < m'_j < m'_i} q_\lambda(t_{m_1} - t_{m'_j}, k_{m_1} k_{m_i})$ in (34) arise from the products

$$\prod_{m_i < m'_j} q_\lambda(t - t_{m'_j}, k k_{m_i}) \prod_{m'_i < m'_j} q_\lambda^{-1}(t - t_{m'_j}, k k_{m'_i})$$

in (31) due to cancellation of corresponding terms because of δ-functions $\delta(k_{m_i} - k_{m'_i})$ in the correlator (32) for $N = 2n - 2$. We have

$$\begin{aligned} & \prod_{m_i < m'_j} q_\lambda(t - t_{m'_j}, k k_{m_i}) \prod_{m'_i < m'_j} q_\lambda^{-1}(t - t_{m'_j}, k k_{m'_i}) \\ & \quad = \prod_{m'_i < m'_j} q_\lambda(t - t_{m'_j}, k k_{m_i}) \prod_{m_i < m'_j < m'_i} q_\lambda(t - t_{m'_j}, k k_{m_i}) \prod_{m'_i < m'_j} q_\lambda^{-1}(t - t_{m'_j}, k k_{m_i}) \\ & \quad = \prod_{m_i < m'_j < m'_i} q_\lambda(t - t_{m'_j}, k k_{m_i}). \end{aligned}$$

Let us prove now that the (34) is equal in fact to (32). This will give a proof of the theorem.

We have

$$\begin{aligned} & \prod_{m_i < m'_j} q_\lambda^{-1}(t_{m_1} - t_{m_i}, k_{m_1} k_{m_i}) \prod_{m'_i < m'_j} q_\lambda(t_{m_1} - t_{m'_i}, k_{m_1} k_{m'_i}) \\ & \quad = \prod_{m'_i < m'_j} q_\lambda^{-1}(t_{m_1} - t_{m_i}, k_{m_1} k_{m_i}) \prod_{m_i < m'_j < m'_i} q_\lambda^{-1}(t_{m_1} - t_{m_i}, k_{m_1} k_{m_i}) \\ & \quad\quad \times \prod_{m'_i < m'_j} q_\lambda(t_{m_1} - t_{m'_i}, k_{m_1} k_{m'_i}) \end{aligned} \tag{35}$$

because $m_i < m'_i$. From (32) for $2n - 2$ we have $k_{m_i} = k_{m'_i}$. By using this and combining the first product with the third we obtain

$$\prod_{m'_i < m'_j} q_\lambda(t_{m_i} - t_{m'_i}, k_{m_1} k_{m_i}) \prod_{m_i < m'_j < m'_i} q_\lambda^{-1}(t_{m_1} - t_{m_i}, k_{m_1} k_{m_i}). \tag{36}$$

Using the change of variables in the second product in (36)

$$t_{m_1} - t_{m_i} = (t_{m_1} - t_{m'_j}) - (t_{m'j} - t_{m_i})$$

and the property $k_{m_1} = k_{m'_j}$ we find that (35) equals

$$\prod_{m'_i<m'_j} q_\lambda(t_{m_i} - t_{m'_i}, k_{m_1}k_{m_i}) \prod_{m_i<m'_j<m'_i} q_\lambda^{-1}(t_{m_1} - t_{m'_j}, k_{m_1}k_{m_i}) \prod_{m_i<m'_j<m'_i} q_\lambda(t_{m_i} - t_{m'_j}, k_{m'_j}k_{m_i}).$$

Substituting this into equation (34) we obtain

$$\sum_{j=1}^{n} \delta(k_{m_1} - k_{m'_j}) \frac{1}{\lambda^2} q_\lambda(t_{m_1} - t_{m'_j}, \tilde{\omega}(k_{m_1}) + k_{m_1}p) \prod_{m'_i<m'_j} q_\lambda(t_{m_i} - t_{m'_i}, k_{m_1}k_{m_i})$$
$$\times \prod_{m_i<m'_j<m'_i} q_\lambda(t_{m_i} - t_{m'_j}, k_{m'_j}k_{m_i}) \langle \hat{a}_\lambda(t_{m_1}, k_{m_1}) \ldots \hat{a}_\lambda^\dagger(t_{m'_j}, k_{m'_j}) \ldots \rangle. \quad (37)$$

Here the notion $\langle \ldots \hat{a} \ldots \rangle$ means that we omit the operator $\hat{a}$ in this correlation function. For the $(2n-2)$-point correlator in (37) we use the formula (32) for $\varepsilon - \{m_1, m'_j\}$:

$$\langle \hat{a}_\lambda(t_{m_1}, k_{m_1}) \ldots \hat{a}_\lambda^\dagger(t_{m'_j}, k_{m'_j}) \ldots \rangle$$
$$= \sum_{\sigma(\varepsilon - \{m_1, m'_j\})} \prod_{h=1}^{n-1} \delta(k_{n_h} - k_{n'_h})$$
$$\times \frac{1}{\lambda^2} q_\lambda \left((t_{n_h} - t_{n'_h}), \left(\tilde{\omega}(k_{n_h}) + k_{n_h} p + \sum_{n_\alpha<n_h<n'_\alpha} k_{n_\alpha} \cdot k_{n_h} \right) \right)$$
$$\times \prod_{(n_j,n'_j),(n_i,n'_i);i,j=1,\ldots,n-1:n_j<n_i<n'_j<n'_j} q_\lambda(t_{n_i} - t_{n'_j}, k_{n_i} \cdot k_{n'_j}) \quad (38)$$

where $\sigma(\varepsilon - \{m_1, m'_j\}) = \{(n_j < n'_j) : j = 1, \ldots, n-1\}$ is a partition (without one pair) of $\{1, \ldots, 2n\}$ associated with $\varepsilon - \{m_1, m'_j\}$. The indices n_h correspond to annihilators, n'_h correspond to creators.

Substituting (38) into (37) we obtain

$$\sum_{j=1}^{n} \delta(k_{m_1} - k_{m'_j}) \frac{1}{\lambda^2} q_\lambda(t_{m_1} - t_{m'_j}, \tilde{\omega}(k_{m_1}) + k_{m_1}p) \prod_{m'_i<m'_j} q_\lambda(t_{m_i} - t_{m'_i}, k_{m_1}k_{m_i})$$
$$\times \prod_{m_i<m'_j<m'_i} q_\lambda(t_{m_i} - t_{m'_j}, k_{m'_j}k_{m_i}) \sum_{\sigma(\varepsilon - \{m_1, m'_j\})} \prod_{h=1}^{n-1} \delta(k_{n_h} - k_{n'_h})$$
$$\times \frac{1}{\lambda^2} q_\lambda \left((t_{n_h} - t_{n'_h}), \left(\tilde{\omega}(k_{n_h}) + k_{n_h} p + \sum_{n_\alpha<n_h<n'_\alpha} k_{n_\alpha} \cdot k_{n_h} \right) \right)$$
$$\times \prod_{(n_j,n'_j),(n_i,n'_i);i,j=1,\ldots,n-1:n_j<n_i<n'_j<n'_j} q_\lambda(t_{n_i} - t_{n'_j}, k_{n_i} \cdot k_{n'_j}). \quad (39)$$

It is easy to see that

$$\sum_{j=1}^{n} \sum_{\sigma(\varepsilon - \{m_1, m'_j\})} = \sum_{\sigma(\varepsilon)}. \quad (40)$$

Using (40) and combining the first product in (39) with the third and the second product with the fourth we obtain (32).

This finishes the proof of the theorem. □

Acknowledgments

This work was partially supported by INTAS 96-0698 and RFFI-9600312 grants. IVV is supported in part by a fellowship of the Italian Ministry of Foreign Affairs organized by the Landau Network-Centro Volta.

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